MATH 2250 Calculus I
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Name: Answers
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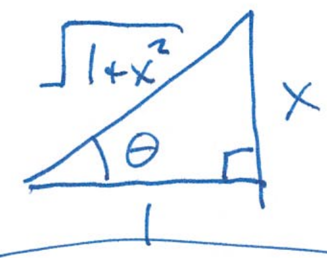
Practice Test No. 2

Show all of your work, label your answers clearly, and do not use a calculator.

Problem 1 (15 points) Derive the derivatives of the given inverse trig functions *using reference triangles*:

a $\frac{d}{dx}\arctan(x) = \frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{\sec^2(\tan^{-1}(x))}$ (Using our formula for $\frac{d}{dx}(f^{-1}(x))$)

Call $\tan^{-1}(x) = \theta$, so that $x = \tan(\theta) = \frac{\text{opp.}}{\text{adj.}}$

$\Rightarrow$ 

And so $\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{\sec^2(\tan^{-1}(x))}$

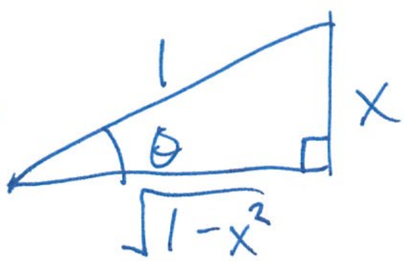
$= \frac{1}{\sec^2(\theta)} = \cos^2(\theta) = \left(\frac{\text{adj.}}{\text{hyp.}}\right)^2$

$= \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}$

a $\frac{d}{dx}\arcsin(x) = \frac{d}{dx}\left(\sin^{-1}(x)\right)$

$= \frac{1}{\cos(\sin^{-1}(x))}$ (Using our formula for $\frac{d}{dx}(f^{-1}(x))$)

$= \frac{1}{\cos(\theta)}$ $\Rightarrow$ $x = \sin(\theta) = \frac{\text{opp.}}{\text{hyp.}}$

$\Rightarrow$ 

So $\frac{1}{\cos(\theta)} = \frac{1}{\frac{\text{adj.}}{\text{hyp.}}} = \frac{\text{hyp.}}{\text{adj.}}$

$= \frac{1}{\sqrt{1-x^2}}$

$\rightarrow \frac{d}{dx}\left(\arcsin(x)\right) = \frac{1}{\sqrt{1-x^2}}$

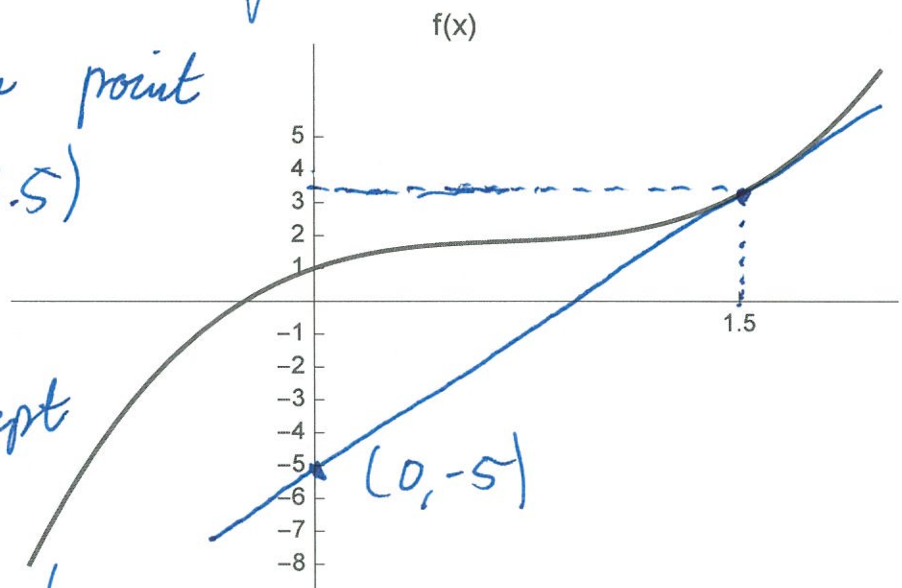
Problem 2 (10 points) For full credit on this problem, make sure that you show your work by showing me how you are using the graph **and** by writing a sentence explaining what you are doing. (You should be drawing a line somewhere on the graph).

a Estimate $f(1.5)$

Looking at the graph, $f(1.5) \approx 3.5$

b Estimate $f'(1.5)$

Draw a tangent line at $x = 1.5$ and then use the point $(1.5, 3.5)$ and the y-intercept of the tangent line to estimate the slope of the tangent line



$$\Rightarrow f'(1.5) \approx \frac{3.5 - (-5)}{1.5 - 0} = \frac{8.5}{1.5}$$

$$= 8.5\left(\frac{2}{3}\right) = \left(\frac{17}{2}\right)\left(\frac{2}{3}\right) = \frac{17}{3}$$

Problem 3 (10 points) Find the following derivatives using derivative rules:

a $f(x) = 4x^3 + 2x^{-1}$, $f'(x) =$ $12x^2 - 2x^{-2}$

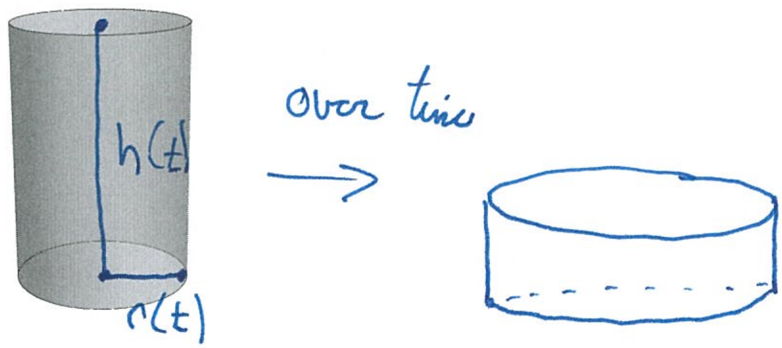
b $g(x) = e^x(5x^2 + x)$, $g'(x) =$ $e^x(5x^2 + x) + e^x(10x + 1)$

c $h(x) = 2\dfrac{\sin(x)}{\ln(x)}$, $h'(x) =$ $2\left(\dfrac{\ln(x)\cos(x) - \sin(x)\left(\frac{1}{x}\right)}{(\ln(x))^2}\right)$

d $y(x) = 2(3x + 50)^{2015}$, $y'(x) =$ $4030(3x + 50)^{2014}(3)$

e $u(x) = \dfrac{\overset{x}{\cancel{\ln(e^x)}} + \ln(\ln(x))}{5}$, $u'(x) =$ $\frac{1}{5}\left(1 + \frac{1}{\ln(x)} \cdot \frac{1}{x}\right)$

Problem 4 (20 points) A manufacturing plant receives cylindrical ingots of steel and then compresses them to make flattened steel sheets. The radius of the of the **cylindrical** ingots it receives is $10cm$ and the height of the **cylindrical** ingots is $25cm$. The ingots are heated, and then the press the plant uses to flatten the ingots crushes them so that their height changes at a constant $-2cm/min$. Assuming that the ingots stay **cylindrical** as they are being crushed, at what rate is the radius of the ingots changing when the height of the ingots is equal to $5cm$? (*Hint: the volume of the ingots as they are being crushed stays constant.*)



What we are given: Initially, i.e. at $t=0$, $r(0)=10$ and $h(0)=25$

$h'(t) = -2$

V is constant, so $V = \pi(10^2)(25) = 2500\pi$

At any time t, $V = 2500\pi = \pi(r(t))^2 h(t)$

$\Rightarrow 0 = \pi\left(2r(t)r'(t)h(t) + (r(t))^2 h'(t)\right)$

Let t^* be the time when $h(t^*) = 5$, so we need to solve for $r(t^*)$ still. $2500\pi = \pi(r(t^*))^2 h(t^*)$

$\Rightarrow \frac{2500}{5} = (r(t^*))^2 \Rightarrow r(t^*) = \sqrt{500} = 10\sqrt{5}$

$\Rightarrow 0 = 2r(t^*)r'(t^*)h(t^*) + (r(t^*))^2 h'(t^*)$

$\Rightarrow r'(t^*) = -(r(t^*))^2 h'(t^*)/2r(t^*)h(t^*) = -500(-2)/2(10\sqrt{5})(5)$

Problem 5 (15 points) Given the equation $y^4 = y^2 - x^2$:

a Use implicit differentiation to show that $y'(x) = \dfrac{-x}{2y^3 - y}$

$$(y(x))^4 = (y(x))^2 - x^2$$

$$4(y(x))^3 y'(x) = 2y(x)y'(x) - 2x$$

$$4(y(x))^3 y'(x) - 2y(x)y'(x) = -2x$$

$$y'(x)\left(4(y(x))^3 - 2y(x)\right) = -2x$$

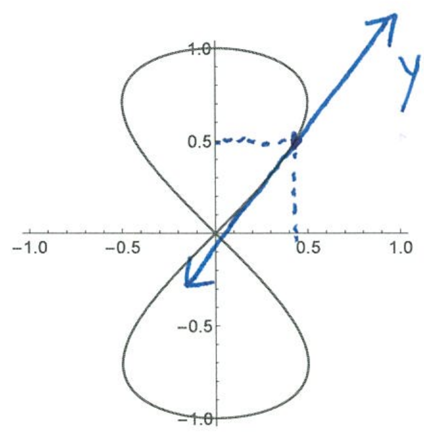
$$\Rightarrow y'(x) = \frac{-2x}{4(y(x))^3 - 2y(x)} = \frac{-x}{2y^3 - y}$$

b Find the tangent line at the point $\left(\dfrac{\sqrt{3}}{4}, \dfrac{1}{2}\right)$

$$y - \frac{1}{2} = \left(\frac{-\frac{\sqrt{3}}{4}}{2\left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)}\right)\left(x - \frac{\sqrt{3}}{4}\right)$$

$$\Rightarrow y = \sqrt{3}\left(x - \frac{\sqrt{3}}{4}\right) + \frac{1}{2}$$

c Given the curve that satisfies the equation $y^4 = y^2 - x^2$, on the same graph, sketch the tangent line you just found.



$y = \sqrt{3}\left(x - \frac{\sqrt{3}}{4}\right) + \frac{1}{2}$

Problem 6 (20 points) Astronomers estimate that on a certain exo-planet, a rock thrown upwards from $10m$ above the surface with an initial velocity of $12m/sec$ would have a height above the surface given by the equation $s(t) = -14t^2 + 12t + 10$

a Find the rock's velocity and acceleration as function of time t.

$v(t) =$ $-28t + 12$ $\qquad\qquad$ $a(t) =$ -28

b How long does it take the rock to reach its highest point?

Set $v(t) = 0$ and solve for t

$\Rightarrow \quad 0 = -28t + 12 \Rightarrow 28t = 12 \Rightarrow t = \frac{12}{28} = \frac{3}{7}$

c How high does the rock go?

Plug in the t value from above into $s(t)$

$\Rightarrow \quad s\left(\frac{3}{7}\right) = -14\left(\frac{3}{7}\right)^2 + 12\left(\frac{3}{7}\right) + 10$

d How long does it take the rock to reach half its maximum height?

Set $\frac{1}{2}s\left(\frac{3}{7}\right) = s(t)$ and solve for t

(Don't bother simplifying)

e How long is the rock aloft?

Set $s(t) = 0$ and solve for t

$-14t^2 + 12t + 10 = 0$

$$\Rightarrow \quad t = \frac{-12 \pm \sqrt{144 - 4(-14)(10)}}{2(-14)}$$

Problem 7 (10 points) Use the technique of linearization to estimate the value of $\sqrt[3]{9}$.

Use $f(x) = \sqrt[3]{x}$ and $a = 8$ (because $\sqrt[3]{8} = 2$), so we have $f'(x) = \frac{1}{3}x^{-2/3}$

$$L(x) = f'(a)(x-a) + f(a)$$
$$= \frac{1}{3}(8)^{-2/3}(x-8) + \sqrt[3]{8}$$
$$= \frac{1}{3(\sqrt[3]{8})^2}(x-8) + 2$$
$$= \frac{1}{3(4)}(x-8) + 2$$
$$= \frac{1}{12}(x-8) + 2$$

Now we use $L(9) = \frac{1}{12}(9-8) + 2$

$$= \frac{1}{12} + 2 \approx 2.08333$$

(The actual value is $\sqrt[3]{9} = 2.08008$